big matrix: find row, column, nullspace bases will be on test

- order of basis vectors matters

$B: \left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\} = x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}1\\0\end{bmatrix} \neq_B \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$

• change of basis matrix
- from st. matrix to diff. matrix

No determinants

To Do:
- practice tests,
- office hours
- written HW
- webwork
- remake notesheet 2.0

for subspace: give as span (w)
for basis: give as {vec}

$Ax = Bx \Rightarrow$ rewrite as a nullspace:
$Ax - Bx = 0$
$\vec{x}(A - B) = \vec{0}$
$Ax = Bx$ = nullspace for $(A - B)$ is nullsp

11/13/24 Lecture Notes

- Exam topics
  - Linear Transformations (definition/proofs?)
    - * invertibility, 1-1, onto, ker, range
  - Matrices
    - * operations, inverses, construction w/ LT's
  - Subspaces
    - * Definition, basis, dimension, row/col, null spaces, rank-nullity
  - change of basis

Ex) A is a rank 2 matrix with $\begin{bmatrix}-1\\1\\0\\2\end{bmatrix}$ in null(A)

A must be $n \times 4$ in order to multiply by $\mathbb{R}^4$ vector

Smallest # of rows: 2 (by range 2)

A vector in the nullspace ≠ only vector in nullspace

A is 2×4, nullity(A) = 2, rank(A) = 2

$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \end{bmatrix} \Rightarrow A\begin{bmatrix}-1\\1\\0\\2\end{bmatrix} = \vec{0} \Rightarrow \left[\begin{array}{cccc|c} -1 & 1 & 0 & 2 & 0 \end{array}\right] \Rightarrow$ 3 free variables

$x_1 = s_1 + 2s_3,\ x_2 = s_3,\ x_3 = s_2,\ x_4 = s_1$

$A = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{2x_1 + x_4 = 0} \Rightarrow \text{null}(A) = \left\{ s_1\begin{bmatrix}0\\1\\0\\0\end{bmatrix} + s_2\begin{bmatrix}-1/2\\0\\0\\1\end{bmatrix} : s_1 \in \mathbb{R},\ s_2 \in \mathbb{R} \right\} = \text{span}\left(\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1/2\\0\\0\\1\end{bmatrix}\right)$

$2x_1 + x_4 = 0 \Rightarrow x_1 = -\frac{1}{2}s_2,\ x_2 = s_1,\ x_3 = 0,\ x_4 = s_2$ (I think, double check)

- Nullspace (matrix) = Kernel (linear transformation)

Ex) $T: \mathbb{R}^5 \to \mathbb{R}^3$

$\dim(\ker(T)) \geq 2 \Rightarrow$ because rank(A) + nullity(A) = # cols

dim(range(T)) + dim(ker(T)) = dim(domain) = m
= 2

$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ 2x_2 + x_3 \\ x_5 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$

$x_1 = x_2 = -\frac{1}{2}s_2,\ x_3 = s_2$

$x_2 = -\frac{1}{2}s_2,\ x_4 = s_1,\ x_5 = 0$

rank(T) = 3

range(T) = $\mathbb{R}^3$ (✓) $= \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \ldots\right)$

basis for span $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right)$

$\ker(T) = \text{span}\left(\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1/2\\-1/2\\1\\0\\0\end{bmatrix}\right) \Rightarrow s_1\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} + s_2\begin{bmatrix}-1/2\\-1/2\\1\\0\\0\end{bmatrix}$

Basis of $\ker(T) = \left\{\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-1/2\\-1/2\\1\\0\\0\end{bmatrix}\right\}$

Is T onto? Yes! Is T 1-1? No!

Ex) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$, Let $V = \{\vec{x} \text{ in } \mathbb{R}^3 \mid A\vec{x} = B\vec{x}\}$

(a) $A\vec{0} = B\vec{0} \Rightarrow \vec{0}$ satisfies condition ✓

(b) Let u, v be in V ⇒ $au = bu$, $av = bv \Rightarrow A(u+v) = Au + Av = Bu + BV$ ✓

(c) $A(c\vec{u}) = c(Au) = cBu = Bc\vec{u} \Rightarrow$ ✓

Yes V is a subspace.